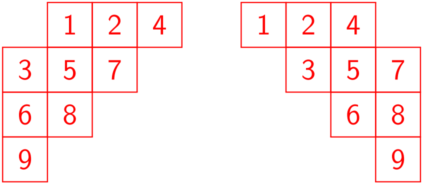
What is
a standard Young tableau?

Ron Adin

Department of Mathematics
Bar-Ilan University

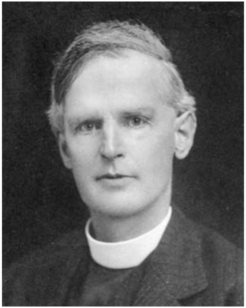
"What is . . .?" Seminar
Bar-Ilan University, 10 Kislev 5776



Abstract

More than a hundred years ago, Frobenius and Young based the emerging representation theory of the symmetric group on the combinatorial objects now called Standard Young Tableaux (SYT). Many important features of these classical objects have since been discovered, including some surprising interpretations and the celebrated hook length formula for their number.
In recent years, SYT of non-classical shapes have come up in research and were shown to have, in many cases, surprisingly nice enumeration formulas.
The talk will present some gems from the study of SYT over the years, including some exciting recent progress. It is partially based on a survey chapter, joint with Yuval Roichman, in the recent CRC Handbook of Combinatorial Enumeration.

Founders

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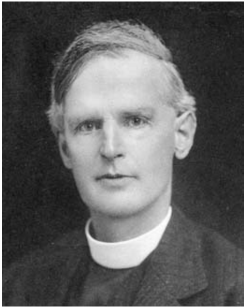

A. Young

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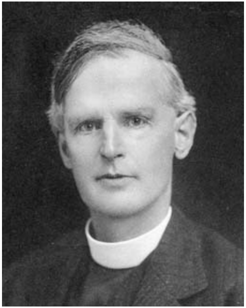

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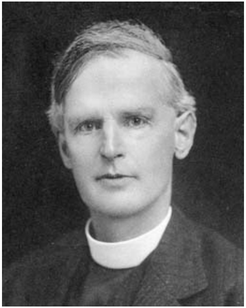

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F. G. Frobenius

P. A. MacMahon

Classical

Introduction

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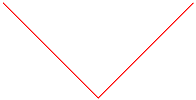

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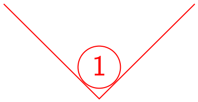

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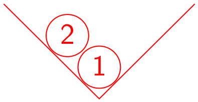

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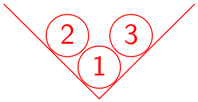

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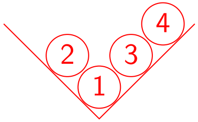

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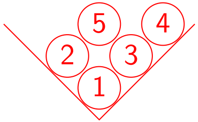

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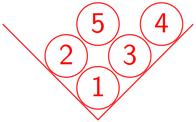

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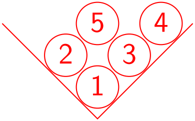

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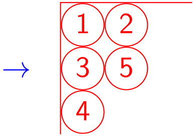

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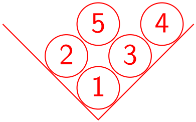

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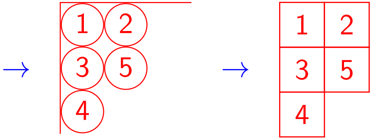
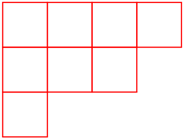


Diagrams and Tableaux

Diagrams and Tableaux

partition $\longleftrightarrow$ diagram/shape

$\lambda = (4, 3, 1) \vdash 8$

$[\lambda] =$ 

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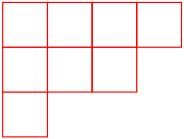


Standard Young Tableau (SYT):

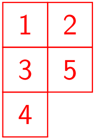
$T =$

1	2	5	8
3	4	6	
7			

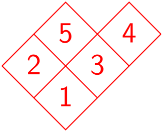
$\in \mathrm{SYT}(4, 3, 1).$

Entries increase along rows and columns

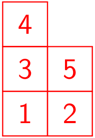
Conventions



English



Russian



French

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4	5	

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$$\lambda = (3, 2), \quad f^{\lambda} = 5$$

SYT and S_n Representations

$S_n =$ the symmetric group on n letters

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Corollary: (regular representation)

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

RS(K) Correspondence

[Robinson, Schensted (, Knuth)]

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$$\pi \longleftrightarrow (P, Q)$$

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4236517 $\longleftrightarrow$ (P , Q):

1	3	5	7
2	6		
4			

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$$4236517 \longleftrightarrow \left(\begin{array}{cccc} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{array} \;,\; \begin{array}{cccc} 1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \end{array} \right)$$

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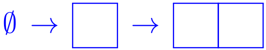

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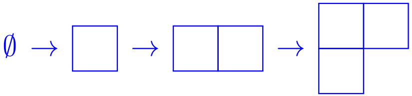

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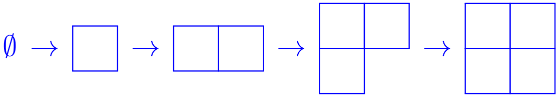

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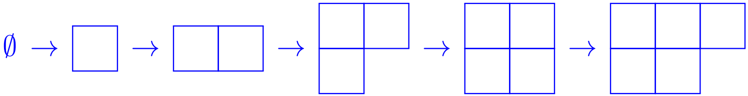

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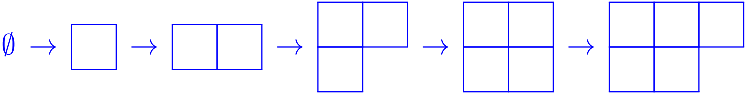

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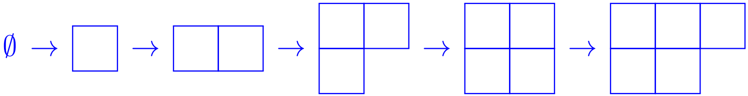

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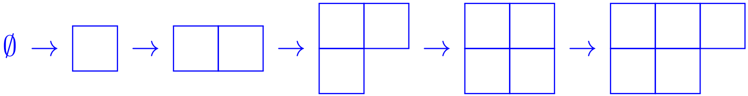

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$$\mathrm{SYT}(\lambda) \quad \longleftrightarrow \quad \text{maximal chains in the Young lattice from } \emptyset \text{ to } \lambda$$

The number of such maximal chains is therefore f^λ.

Interpretation 2: Lattice Paths

Each SYT of shape $\lambda = (\lambda_1, \ldots, \lambda_t)$ corresponds to a lattice path in $\mathbb{R}^t$, from the origin 0 to the point λ, where in each step exactly one of the coordinates changes (by adding 1), while staying within the region

$$\{(x_1, \ldots, x_t) \in \mathbb{R}^t \mid x_1 \geq \ldots \geq x_t \geq 0\}.$$

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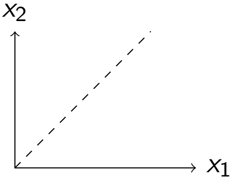

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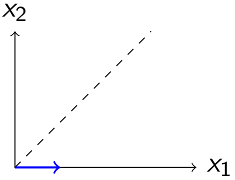

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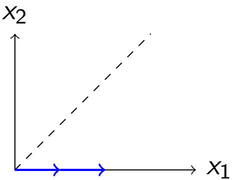

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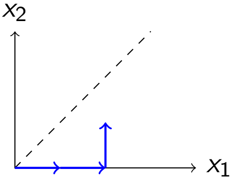

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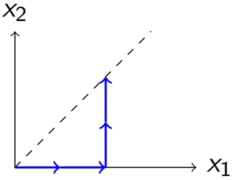

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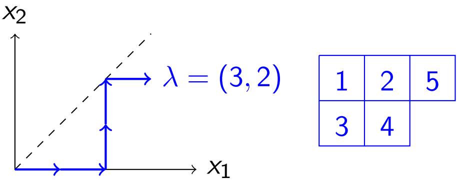

Interpretation 3: Order Polytope

The order polytope corresponding to a diagram D is

$$P(D) := \{f : D \to [0,1] \,|\, c \leq_D c' \implies f(c) \leq f(c') \, (\forall c, \, c' \in D)\},$$

where $\leq_D$ is the natural partial order between the cells of D. It is a closed convex subset of the unit cube $[0,1]^D$.

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d	e	

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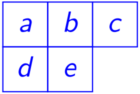

$$f : \{a, b, c, d, e\} \to [0,1]$$
$$f(a) \leq f(b) \leq f(c)$$
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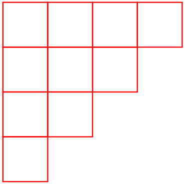
Observation:

$$\operatorname{vol} P(D) = \frac{f^D}{|D|!}.$$

Interpretation 4: Reduced Words (1)

The following theorem was conjectured and first proved by Stanley using symmetric functions. Edelman and Greene later provided a bijective proof.

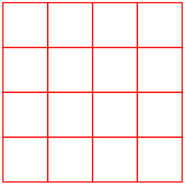
Theorem: [Stanley 1984, Edelman-Green 1987]
The number of reduced words (in adjacent transpositions) of the longest permutation $w_0 := [n, n-1, ..., 1]$ in S_n is equal to the number of SYT of staircase shape $\delta_{n-1} = (n-1, n-2, ..., 1)$.



Interpretation 4: Reduced Words (2)

An analogue for type B (signed permutations) was conjectured by Stanley and proved by Haiman.

Theorem: [Haiman 1989]
The number of reduced words (in the alphabet of Coxeter generators) of the longest element $w_0 := [-1, -2, ..., -n]$ in B_n is equal to the number of SYT of square $n \times n$ shape.



Formulas: Product and Determinant

For a partition $\lambda = (\lambda_1, \ldots, \lambda_t)$, let

$$\ell_i := \lambda_i + t - i \qquad (1 \leq i \leq t).$$

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Theorem: [Frobenius 1900, MacMahon 1909, Young 1927]

$$f^\lambda = \frac{|\lambda|!}{\prod_{i=1}^{t} \ell_i!} \cdot \prod_{(i,j):\, i<j} (\ell_i - \ell_j).$$

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Theorem: (Determinantal Formula)

$$f^\lambda = |\lambda|! \cdot \det \left[\frac{1}{(\lambda_i - i + j)!} \right]_{i,j=1}^t ,$$

using the convention $1/k! := 0$ for negative integers k.

Hook Length Formula

The hook length of a cell $c = (i, j)$ in a diagram of shape λ is

$$h_c := \lambda_i + \lambda'_j - i - j + 1.$$

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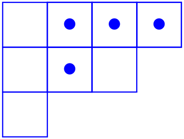
$$h_c := \lambda_i + \lambda'_j - i - j + 1.$$



hook of $c = (1, 2)$

6	4	3	1
4	2	1	
1			

hook lengths

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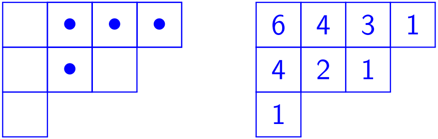


hook of $c = (1, 2)$ hook lengths

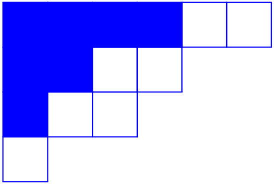
Theorem: [Frame-Robinson-Thrall, 1954]

$$f^\lambda = \frac{|\lambda|!}{\prod_{c \in [\lambda]} h_c}.$$

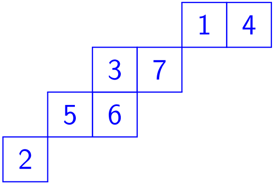
Still Classical

Skew Shapes

If λ and μ are partitions such that $[\mu] \subseteq [\lambda]$, namely $\mu_i \leq \lambda_i$ $(\forall i)$, then the skew diagram of shape λ/μ is the set difference $[\lambda/\mu] := [\lambda] \setminus [\mu]$ of the two ordinary shapes.



$$= \quad [(6,4,3,1)/(4,2,1)]$$



$$\in \quad \mathrm{SYT}((6,4,3,1)/(4,2,1)).$$

Skew Shapes and Representations

λ/μ skew shape of size n	$\longrightarrow$	$\chi^{\lambda/\mu}$ (reducible) character of S_n
SYT(λ/μ)	$\longleftrightarrow$	basis of representation space
$f^{\lambda/\mu}$	$=$	$\chi^{\lambda/\mu}(id)$

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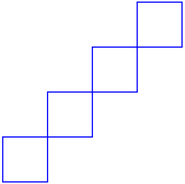
For example,



$\longleftrightarrow$ the regular character

$$\chi^{\text{reg}}(g) = |G|\delta_{g,id}$$

$$(G = S_4)$$

Skew Determinantal Formula

Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be partitions such that $\mu_i \le \lambda_i$ $(\forall i)$.

Theorem [Aitken 1943, Feit 1953]

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det\left[\frac{1}{(\lambda_i - \mu_j - i + j)!}\right]_{i,j=1}^{t},$$

with the conventions $\mu_j := 0$ for $j > s$ and $1/k! := 0$ for negative integers k.

Unfortunately, no product or hook length formula is known for general skew shapes.

Shifted Shapes

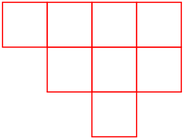
A partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ is called strict if its parts λ_i are strictly decreasing: $\lambda_1 > \ldots > \lambda_t > 0$.

Shifted Shapes

A partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ is called strict if its parts λ_i are strictly decreasing: $\lambda_1 > \ldots > \lambda_t > 0$.
The shifted diagram of shape λ is the set

$$D = [\lambda^*] := \{(i,j) \,|\, 1 \leq i \leq t,\ i \leq j \leq \lambda_i + i - 1\}.$$

Note that $(\lambda_i + i - 1)_{i=1}^{t}$ are weakly decreasing.

$\lambda = (4,3,1) \quad \Longrightarrow \quad [\lambda^*] =$ 

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & 8 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} \in \mathrm{SYT}((4,3,1)^*).$$

Shifted Shapes and Representations

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Strict partitions λ of n essentially correspond to irreducible projective characters of S_n.

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$$g^\lambda := \# \mathsf{SYT}(\lambda^*)$$

Corollary:

$$\sum_{\lambda \models n} 2^{n-t} (g^\lambda)^2 = n!$$

Shifted Formulas

Like ordinary shapes, the number g^λ of SYT of shifted shape λ has three types of formulas – product, determinantal and hook length.

Theorem [Schur 1911, Thrall 1952]

$$g^\lambda = \frac{|\lambda|!}{\prod_{i=1}^{t} \lambda_i!} \cdot \prod_{(i,j):\, i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}$$

Theorem

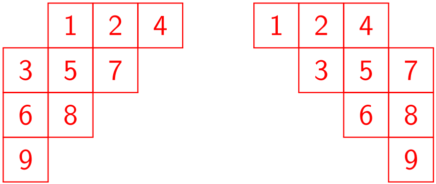
$$g^\lambda = \frac{|\lambda|!}{\prod_{(i,j):\, i<j}(\lambda_i + \lambda_j)} \cdot \det\left[\frac{1}{(\lambda_i - t + j)!}\right]_{i,j=1}^{t}$$

Theorem

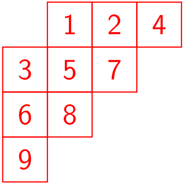
$$g^\lambda = \frac{|\lambda|!}{\prod_{c\in[\lambda^*]} h_c^*}$$

Non-Classical

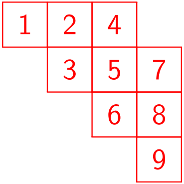
Truncated Shapes



Truncated Shapes

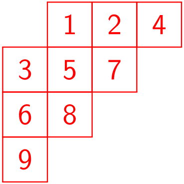


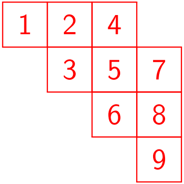
classical



non-classical

Truncated Shapes

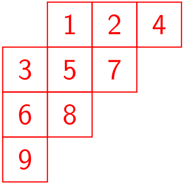




classical
skew

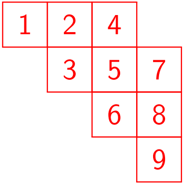
non-classical
shifted, truncated

Truncated Shapes



classical
skew

$\# \,\text{SYT} = 768$



non-classical
shifted, truncated

$\# \,\text{SYT} = 4$

Truncated Shifted Staircase

The number of SYT whose shape is a shifted staircase with a truncated corner came up in a combinatorial setting, counting the number of geodesics (shortest paths) between antipodes in a certain flip graph (of triangulations) [AFR 2010].

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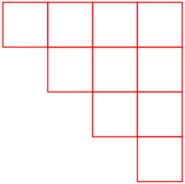
$$\lambda = (9,9,8,7,6,5,4,3,2,1)$$
$$N = 54 \text{ (size)}$$

$$g^\lambda = 116528733315142075200$$
$$= 2^6 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53$$

The largest prime factor is $< N$!!!

Shifted Staircase (Classical)

Let $\delta_n := (n, n-1, \ldots, 1)$, a shifted staircase shape.



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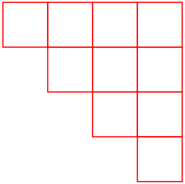


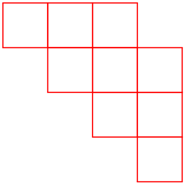
Corollary: (of Schur's product formula for shifted shapes)

$$g^{\delta_n} = N! \cdot \prod_{i=0}^{n-1} \frac{i!}{(2i+1)!},$$

where $N := |\delta_n| = \binom{n+1}{2}$.

Truncated Shifted Staircase

The first example of a truncated shape:



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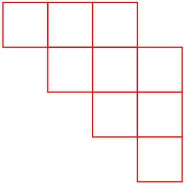


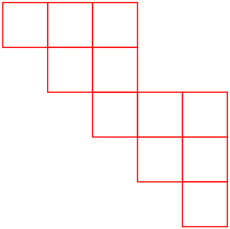
Theorem: [A-King-Roichman '11, Panova '12] The number of SYT of shape $\delta_n \setminus (1)$ is equal to

$$g^{\delta_n} \frac{C_n C_{n-2}}{2\, C_{2n-3}},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Truncated Shifted Staircase

More generally, truncating a square from a shifted staircase shape:

$[\delta_5 \setminus (2^2)] =$ 

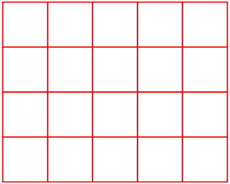
Theorem: [AKR] The number of SYT of truncated shifted staircase shape $\delta_{m+2k} \setminus ((k-1)^{k-1})$ is

$$g^{(m+k+1,\dots,m+3,m+1,\dots,1)} g^{(m+k+1,\dots,m+3,m+1)} \cdot \frac{N!M!}{(N-M-1)!(2M+1)!},$$

where $N = \binom{m+2k+1}{2} - (k-1)^2$ is the size of the shape and $M = k(2m+k+3)/2 - 1$.

Similarly for truncating "almost squares" $(k^{k-1}, k-1)$.

Rectangle (Classical)

$[(5^4)] =$ 

Observation:

The number of SYT of rectangular shape (n^m) is

$$f^{(n^m)} = (mn)! \cdot \frac{F_m F_n}{F_{m+n}},$$

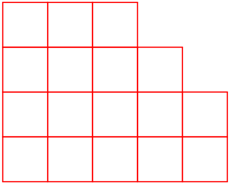
where

$$F_m := \prod_{i=0}^{m-1} i!.$$

Truncated Rectangle

Truncate a rectangle by a (shifted) staircase.

$[(5^4) \setminus \delta_2] =$



Theorem: [Panova]
Let $m \geq n \geq k$ be positive integers. The number of SYT of truncated shape $(n^m) \setminus \delta_k$ is

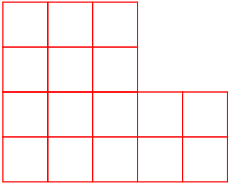
$$\binom{N}{m(n-k-1)} f^{(n-k-1)^m} g^{(m,m-1,\ldots,m-k)} \frac{E(k+1,m,n-k-1)}{E(k+1,m,0)},$$

where $N = mn - \binom{k+1}{2}$ is the size of the shape and $E(r,p,s) = \ldots$.

Truncated Rectangle

Truncate a square from the NE corner of a rectangle:

$[(5^4) \setminus (2^2)] =$



Theorem: [AKR]
The number of SYT of truncated rectangular shape $((n+k)^{m+k}) \setminus (k^k)$ and size N is

$$\frac{N!(mk+m-1)!(nk+n-1)!(m+n)!}{(mk+nk+m+n)!} \cdot \frac{F_{m-1}F_{n-1}F_k}{F_{m+n+k}}.$$

Similar results were obtained for truncation by almost squares.

Truncated Rectangle

The following formula, for a slightly truncated square, was conjectured by AKR and proved by Sun.

Theorem: [Sun '15]
For $n \geq 2$

$$f^{(n^n)\setminus(2)} = \frac{(n^2-2)!(3n-4)!^2 \cdot 6}{(6n-8)!(2n-2)!(n-2)!^2} \cdot \frac{F_{n-2}^2}{F_{2n-4}}.$$

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Theorem: [Snow]
For $n \geq 2$ and $k \geq 0$

$$f^{(n^{k+1})\backslash(n-2)} = \frac{(kn-k)!(kn+n)!}{(kn+n-k)!} \cdot \frac{F_k F_n}{F_{n+k}}.$$

Proof Ideas: Sun

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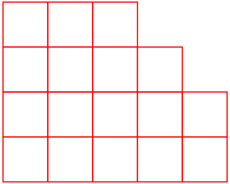
A rectangle truncated by a shifted staircase:



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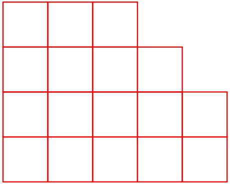

First step:
$\#SYT = N!$ times the volume of the order polytope

Interpretation 3: Order Polytope

The order polytope corresponding to a diagram D is

$$P(D) := \{f : D \to [0,1] \mid c \leq_D c' \Longrightarrow f(c) \leq f(c')\,(\forall c,\, c' \in D)\},$$

where $\leq_D$ is the natural partial order between the cells of D. It is a closed convex subset of the unit cube $[0,1]^D$.

a	b	c
d	e	

$$f : \{a, b, c, d, e\} \to [0,1]$$
$$f(a) \leq f(b) \leq f(c)$$
$$f(d) \leq f(e)$$
$$f(a) \leq f(d)$$
$$f(b) \leq f(e)$$

Observation:

$$\operatorname{vol} P(D) = \frac{f^D}{|D|!}.$$

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$$\int_{0\le t_1\le\ldots\le t_k\le 1}\cdots\int \prod_{i=1}^{k} t_i^{n-k}(1-t_i)^{m-k}\prod_{i<j}(t_j-t_i)\;dt_1\cdots dt_k,$$

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$$\int\cdots\int_{0\le t_1\le\ldots\le t_k\le 1} \prod_{i=1}^{k} t_i^{n-k}(1-t_i)^{m-k}\prod_{i<j}(t_j-t_i)\;dt_1\cdots dt_k,$$

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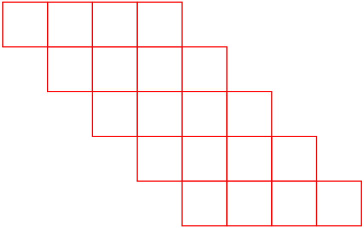
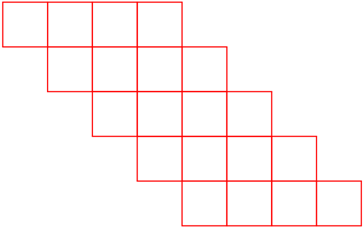
- evaluated using Selberg's integral formula as

$$\prod_{j=0}^{k-1} \frac{\Gamma(n-k+1+j/2)\,\Gamma(m-k+1+j/2)\,\Gamma((j+1)/2)}{\Gamma(n+m-2k+2+(k-1+j)/2)}$$

Shifted Strip



Shifted Strip



Theorem: [Sun]

The number of SYT of truncated shifted shape with n rows and 4 cells in each row is the $(2n-1)$-st Pell number

$$\frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^{2n-1}-(1-\sqrt{2})^{2n-1}\right).$$

There are extensions by Hason.

Open Problems

- Which non-classical shapes have nice/product formulas?
- A modified hook length formula?
- A representation theoretical interpretation?

Thank You!